Name: Solution .
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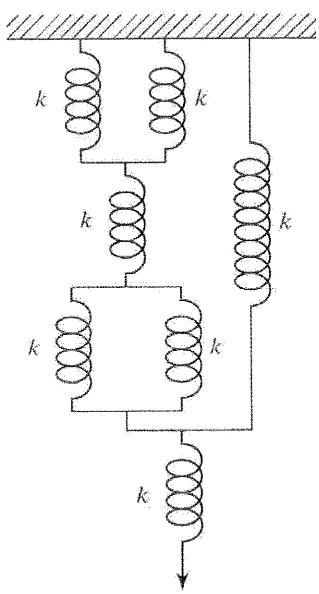
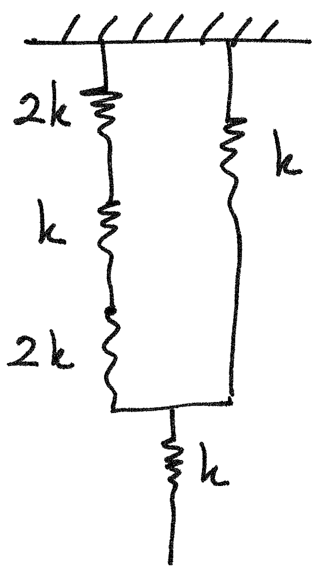
Section: .

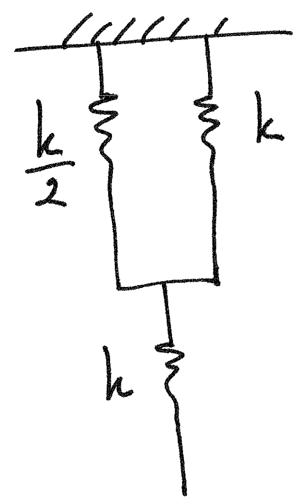
Exam I

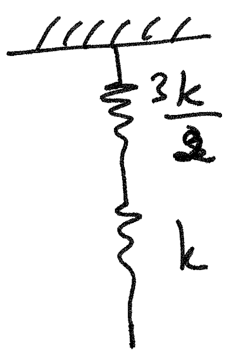
Thursday November 19, 2009
Duration: 1h 30min
All work **must** be shown to receive full credit.

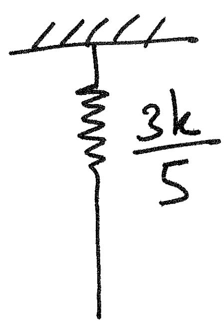
Problem 1. (5 pts.) Determine the equivalent spring constant of the system shown below.











$$k_{eq} = \frac{3k}{5}$$

Problem 2. (10 pts.) A harmonic motion has a frequency of 10 *cps* and its maximum velocity is 4.57 m/s. Determine its **amplitude**, its **period** and its **maximum acceleration**.

$$X(t) = A\sin(\omega t - \phi)$$

$$\dot{x}(t) = A\omega\cos(\omega t - \phi)$$

$$\ddot{X}(t) = -A\omega^2\sin(\omega t - \phi)$$

$f = 10\ Hz \Rightarrow f = \frac{1}{T} \Rightarrow \omega = 2\pi f = 20\pi\ rad/s$

Maximum velocity $v_{max} = A\omega = 4.57\ m/s$

Amplitude : $A = \frac{v_{max}}{\omega} = \frac{4.57}{20\pi} = 0.0727\ m$

Period : $T = \frac{1}{5} = 0.1\,s$

Maximum acceleration : $A\omega^2 = 287.14\ m/s^2$

Problem 3. (20 pts.) A machine is subjected to the motion $x(t) = Acos(50t + \alpha)\ mm$. The initial conditions are given by $x(0) = 3\ mm$ and $\dot{x}(0) = 1000\ mm/s$

a - Find the constants A and α.

b - Express the motion in the form $x(t) = A_1 cos(\omega t) + A_2 sin(\omega t)$.

a- $x(t) = A\cos(50t + \alpha)$

$\dot{x}(t) = -50A\sin(50t + \alpha)$

$t = 0 \Rightarrow \begin{array}{l} x(0) = A\cos\alpha \\ \dot{x}(0) = -50A\sin\alpha \end{array} \Rightarrow \begin{array}{l} A\cos\alpha = x(0) \\ A\sin\alpha = \dfrac{\dot{x}(0)}{-50} \end{array}$

$\Rightarrow A = \sqrt{x(0)^2 + \left(\dfrac{\dot{x}(0)}{50}\right)^2} = \sqrt{3^2 + 20^2} = 20.22\ mm$

$\alpha = \tan^{-1}\left(-\dfrac{\dot{x}(0)/50}{x(0)}\right) = \tan^{-1}\left(-\dfrac{20}{3}\right) = -1.42 = -81.47^\circ$

$\Rightarrow x(t) = 20.22\cos(50t - 1.42)\ mm$

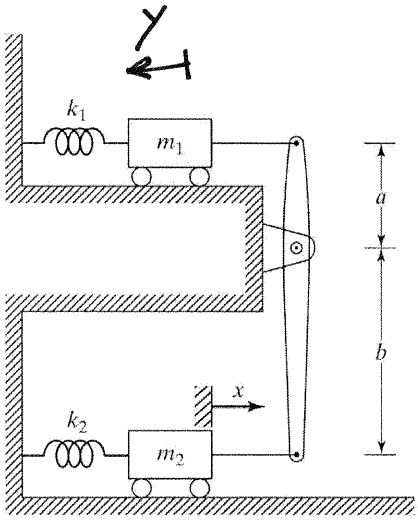
b- $x(t) = A\cos(50t + \alpha) = A\cos 50t\cos\alpha - A\sin 50t\sin\alpha$

$A\cos\alpha = 20.22\cos(-81.47^\circ) \simeq 3\ mm$

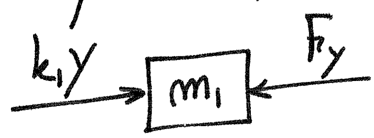
$A\sin\alpha = 20.22\sin(-81.47^\circ) \simeq -20\ mm$

$\Rightarrow x(t) = 3\cos 50t + 20\sin 50t$

Problem 4. (20 pts.) Find the equivalent mass and the equivalent stiffness of the rocker arm assembly below with respect to the x coordinate. Assume that the arm is massless.



• System m_1

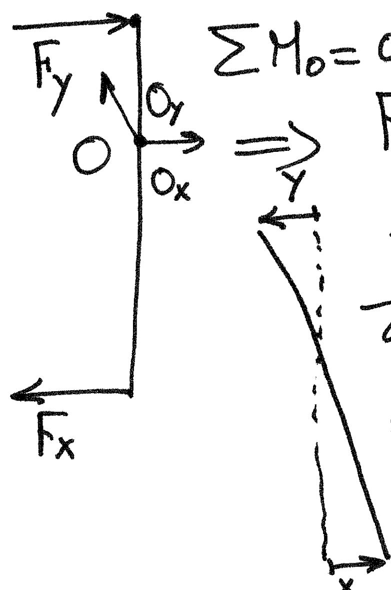


$$\sum F = m_1 \ddot{y}$$

$$\oplus \quad -k_1 y + F_y = m_1 \ddot{y}$$

$$F_y = m_1 \ddot{y} + k_1 y$$

• massless bar



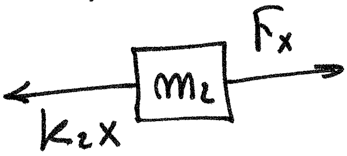
$$\sum M_O = 0 \Rightarrow F_y a + F_x b = 0$$

$$\Rightarrow F_x = -\frac{a}{b} F_y$$

$$\frac{y}{a} = \frac{x}{b}$$

$$\Rightarrow y = \frac{a}{b} x$$

• System m_2



$$\sum F = m_2 \ddot{x}$$

$$\oplus \quad -k_2 x + F_x = m_2 \ddot{x}$$

$$F_x = m_2 \ddot{x} + k_2 x$$

$$\Downarrow$$

$$m_2 \ddot{x} + k_2 x = -\frac{a}{b} F_y = -\frac{a}{b}\left(m_1 \ddot{y} + k_1 y\right) = -\frac{a}{b}\left(m_1 \frac{a}{b}\ddot{x} + k_1 \frac{a}{b} x\right.$$

$$\left(m_2 + m_1 \frac{a^2}{b^2}\right)\ddot{x} + \left(k_2 + k_1 \frac{a^2}{b^2}\right) x = 0$$

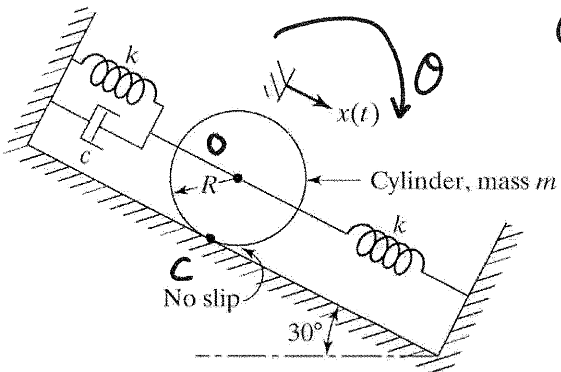
Equivalent mass $\quad m_{eq} = m_2 + m_1 \dfrac{a^2}{b^2}$

Equivalent stiffness $\quad k_{eq} = k_2 + k_1 \dfrac{a^2}{b^2}$

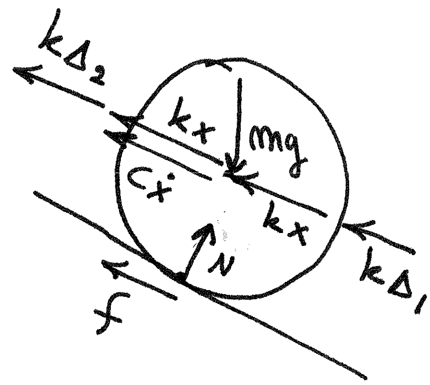
Problem 5. (20 pts.) The cylinder ($I_o = \frac{1}{2}mr^2$) in the figure below rolls without slipping on the inclined plane.

a - Find the natural frequency of the system

b - If $c = \sqrt{3mk}$ whether the system is critically damped, overdamped or underdamped ?



a-



* ignore mg $k\Delta_2$ and $k\Delta_1$

$$\theta \oplus \quad \sum M_c = (I_o + mr^2)\ddot{\theta} \Rightarrow -2kx - c\dot{x} = \frac{3}{2}mr^2\ddot{\theta}$$

$$\Rightarrow \frac{3}{2}mr^2\ddot{\theta} + cr\dot{x} + 2kxr = 0 \qquad x = r\theta$$

$$\Rightarrow \frac{3}{2}m\ddot{x} + c\dot{x} + 2kx = 0$$

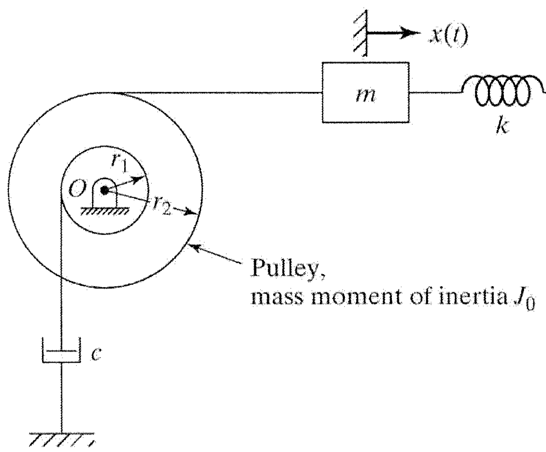
$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{2k}{\frac{3}{2}m}} = 2\sqrt{\frac{k}{3m}}$$

b-
$$\zeta = \frac{c_{eq}}{2\sqrt{m_{eq}k_{eq}}} = \frac{c}{2\sqrt{\frac{3}{2}m\,2k}} = \frac{c}{2\sqrt{3mk}}$$

When $c = \sqrt{3mk} \Rightarrow \zeta = \frac{1}{2}$

$\zeta = \frac{1}{2} < 1 \Rightarrow$ the system is underdamped

Problem 6. (25 pts.) The system shown below has a natural frequency of 5 Hz for the following data: $m = 10\ kg$, $J_o = 5\ kg - m^2$, $r_1 = 10\ cm$, $r_2 = 25\ cm$. When the system is disturbed by giving it an initial displacement, the amplitude of free vibration is reduced by 80% in 10 cycles (i.e. $x_1/x_{11} = 5$). Determine the values of k and c.



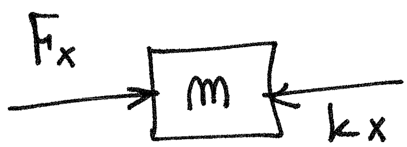
$$\ln\left(\frac{x_1}{x_{11}}\right) = \frac{2\pi \times 10 \times \zeta}{\sqrt{1-\zeta^2}} = \ln(5)$$

$$\Rightarrow 39\,\zeta = \sqrt{1-\zeta^2}$$

$$\Rightarrow \zeta^2 = \frac{1}{1+39^2} = 6.57\times 10^{-4}$$

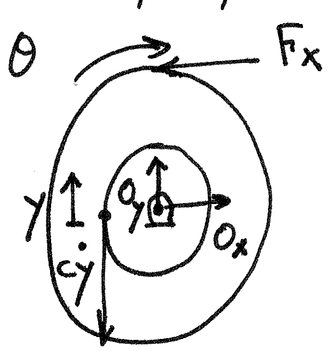
$$\Rightarrow \zeta = 0.0256$$

- m system



$$\underset{\xrightarrow{\oplus}}{\Sigma} F = m\ddot{x} \Rightarrow m\ddot{x} + kx = F_x$$

- Pulley system



$$\oplus \circlearrowleft \Sigma M_o = J_o\ddot{\theta}$$

$$-F_x r_2 - c\dot{y} r_1 = J_o\ddot{\theta}$$

$$x = r_2\theta$$
$$y = r_1\theta \Rightarrow J_o\ddot{\theta} + (m\ddot{x} + kx) r_2 + c\dot{y} r_1 = 0$$

$$\Rightarrow J_o\ddot{\theta} + m r_2^2\ddot{\theta} + k r_2^2\theta + c r_1^2\dot{\theta} = 0$$

$$(J_o + m r_2^2)\ddot{\theta} + c r_1^2\dot{\theta} + k r_2^2\theta = 0$$

$$\omega_n = \sqrt{\frac{k r_2^2}{J_o + m r_2^2}} \qquad f = \frac{\omega_n}{2\pi} = 5\ Hz$$

$$\Rightarrow \omega_n^2 = 4\pi^2 f^2 \Rightarrow \frac{k r_2^2}{J_o + m r_2^2} = 4\pi^2 f^2$$

$$\Rightarrow k = \frac{(J_o + m r_2^2)\, 4\pi f^2}{r_2^2} = 8.8826 \times 10^4\ N/m$$

$$\zeta = \frac{C_{eq}}{2\sqrt{m_{eq} k_{eq}}} = \frac{c\, r_1^2}{2\sqrt{(J_o + m r_2^2)\, k r_2^2}} \Rightarrow$$

$$c = \frac{2\zeta\sqrt{(J_o + m r_2^2)\, k r_2^2}}{r_1^2} = 904.78\ N\text{-}s/m$$

$$k = 8.8826 \times 10^4\ N/m$$

$$c = 904.78\ N\text{-}s/m$$